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THE STABILITY OF NON-LINEAR PULSE SYSTEM IN THE FIRST APPROXIMATION†

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A pulse system, described by a non-linear functional-differential equation, and an "equivalent" continuous non-linear system, obtained from the initial system by replacing the pulse modulator by its static characteristic is considered. It is shown that, for a sufficiently high pulse frequency, asymptotic stability of a state of equilibrium of the pulse system arises from the stability of the equivalent system in the first approximation. © 2003 Elsevier Science Ltd. All rights reserved.

A wide class of non-linear pulse systems is described by the following functional-differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{b}\xi, \quad \xi = M\sigma, \quad \sigma = \mathbf{c}^*\mathbf{x} \tag{1}$$

where $\mathbf{b}$ and $\mathbf{c}$ are constant real m-dimensional column vectors, the asterisk denotes transposition, $\mathbf{f}(\mathbf{x})$ is a continuous m-dimensional vector function and M is a non-linear operator, according to which to each function $\sigma(t)$, continuous in $[0, +\infty)$ there corresponds a function $\xi(t)$ and a sequence $t_n (n = 0, 1, 2, \dots; t_0 = 0)$, possessing the following properties:

positive constants ν_0 and T exist, for which, for all n, the following limit holds

$$\nu_0 T \le t_{n+1} - t_n \le T \tag{2}$$

the function $\xi(t)$ is piecewise-continuous in each interval $[t_n, t_{n+1}]$ and does not change sign in it.

t_n depends only on the values of $\sigma(\tau)$ when $\tau \le t_n$ and $\xi(t)$ depends only on the values of $\sigma(\tau)$ when $\tau \le t$,

a continuous function $\varphi(\sigma)$ exists such that for each n we obtain $\tilde{t}_n \in [t_n, t_{n+1})$, for which the mean value of the nth pulse

$$\upsilon_n = \frac{1}{t_{n+1} - t_n} \int_{t_n}^{t_{n+1}} \xi(t) dt \tag{3}$$

satisfies the relation

$$\upsilon_n = \varphi(\sigma(\tilde{t}_n)) \tag{4}$$

The majority of known forms of pulse modulation (pulse-width modulation, frequency modulation, amplitude modulation, combined modulation, etc. [1–4]) satisfies the above conditions, in which case $\varphi(\sigma)$ is the static characteristic of the pulse modulator (i.e. the dependence of the mean value of the pulse (3) on the modulating signal σ, assuming the latter to be constant).

The simplest example is pulse-width modulation of the first kind (PWN-1), for which $t_n = nT$

$$\xi(t) = \begin{cases} \operatorname{sign}\sigma(nT), & nT \le t < nT + \tau_n \\ 0, & nT + \tau_n \le t < (n+1)T \end{cases} \tag{5}$$

$$\tau_n = TF(|\sigma(nT)|)$$

$F(\lambda)$ is a continuous function, non-decreasing in $(0, +\infty)$, satisfying the conditions $F(0) = 0$, $0 < F(\lambda) \le 1$ when $\lambda > 0$. Pulse-width modulation of the second kind (PWN-2) differs from PWN-1 in that τ_n is calculated not from the last formula of (5) but is the first positive root of the equation

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$$\tau_n = F(|\sigma(nT+\tau_n)|)$$

if such exists in $[0, T]$, and $\tau_n = T$ otherwise. The essential difference between PWN-1 and PWM-2 is the fact that whereas in PWM-1 τ_n is a continuous functional of $\sigma(t)$ for all $\sigma(t) \in C[0, +\infty)$, in the case of PWM-2 this functional is not continuous in the whole space $C[0, +\infty)$. It is obvious that for both forms of pulse-width modulation condition (4) is satisfied when $\varphi(\sigma) = F(|\sigma|)\operatorname{sign}\sigma$, and in the case of PWM-1 $\tilde{t}_n = nT$, while in the case of PWM-2 $\tilde{t}_n = nT + \tau_n$.

Together with (1) we will consider the system of ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{b}\varphi(\sigma), \quad \sigma = \mathbf{c}^*\mathbf{x} \tag{6}$$

which we will call the "equivalent" system. Since the stability of continuous system (6) has been investigated much more than the stability of pulse system (1), we are interested in the hypothesis that, for a sufficiently high pulse frequency (for sufficiently small T), the stability of system (1) follows from the stability of system (6). If the question is stability as a whole, this hypothesis has been disproved in [5, 6] and, using the example of a first-order system with PWM-1, it has been shown that, although the equivalent system is stable as a whole for any values of the parameters, a pulse system can have an infinite set of periodic modes of operation, but not as high as the pulse frequency was. In this paper we prove that this hypothesis holds, if we are dealing with asymptotic stability ("in the small"). If the system of the first approximation [7] is asymptotically stable for continuous system (6), the state of equilibrium $x = 0$ of pulse system (1) will be asymptotically stable.

We will assume that in systems (1) and (6)

$$\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}(\mathbf{x}) \tag{7}$$

where $\mathbf{A}$ is a constant $m \times m$ matrix, while the vector function $\mathbf{a}(\mathbf{x})$ satisfies the condition

$$\|\mathbf{a}(\mathbf{x})\|/\|\mathbf{x}\| \to 0, \quad \|\mathbf{x}\| \to 0 \tag{8}$$

Suppose $\varphi(0) = 0$, the function $\varphi(\sigma)$ is twice continuously differentiable in certain neighbourhood of the point $\sigma = 0$, and the following inequalities are satisfied

$$|\varphi'(\sigma)| \le l, \quad |\varphi''(\sigma)| \le \varphi_+ \tag{9}$$

Suppose λ_- and λ_+ are the minimum and maximum eigenvalues of the matrix $\mathbf{H}$, being a solution of Lyapunov's equation

$$\mathbf{B}^*\mathbf{H} + \mathbf{H}\mathbf{B} = -\mathbf{I}; \quad \mathbf{B} = \mathbf{A} + k\mathbf{b}\mathbf{c}^*; \quad k = \varphi'(0) \tag{10}$$

We introduce the following notation

$$c_1(0) = \frac{24l^2}{\pi^2}\|\mathbf{c}^*\mathbf{A}\|^2,$$

$$c_2(0) = \frac{24l^2}{\pi^2}(|\kappa| + |\kappa_1|T)^2 + 8l^2\kappa^2$$

$$d_1(0) = \frac{4l^2\|\mathbf{c}\|^2 + 2c_1(0)T^2}{1-(4l^2\|\mathbf{c}\|^2 + 2c_2(0))T^2},$$

$$d_2(0) = [c_1(0) + c_2(0)d_1(0)]T^2$$

$$p = \frac{2}{\lambda_-}[\|\mathbf{b}\|^2 d_2(0) + \|\mathbf{A}\mathbf{b} - k\kappa\mathbf{b}\|^2 d_1(0)]$$

$\|\cdot\|$ is the Euclidean norm of a vector or a matrix, and $\kappa = -\mathbf{c}^*\mathbf{b}$, $\kappa_1 = \mathbf{c}^*\mathbf{A}\mathbf{b}$.

Theorem. Suppose conditions (8) and (9) are satisfied, the matrix $\mathbf{B}$ is a Hurwitz matrix, and T is so small that the following inequalities hold

$$4l^2T^2\left[\frac{12}{\pi^2}(|\kappa| + |\kappa_1|T)^2 + 5\kappa^2\right] < 1 \tag{11}$$

$$4\lambda_+^3 pT^2 < 1 \tag{12}$$

$$l|\kappa|T + lT\|\mathbf{c}\|(\|\mathbf{b}\| + T\|\mathbf{Ab}\|)\exp(\|\mathbf{A}\|T) < 1 \tag{13}$$

Then the state of equilibrium $\mathbf{x} = 0$ of system (1) is asymptotically stable.

Remark. When checking the conditions of the theorem it is useful to bear the following inequalities in mind [8]

$$\lambda_+ \le \frac{1}{\min_k|\nu_k(\mathbf{B}^* + \mathbf{B})|}, \quad \lambda_- \ge \frac{1}{2\sqrt{\max_k \mu_k(\mathbf{B}^*\mathbf{B})}}$$

where $\nu_k(\mathbf{B}^* + \mathbf{B})$ and $\mu_k(\mathbf{B}^*\mathbf{B})$ are eigenvalues of the matrices $\mathbf{B}^* + \mathbf{B}$ and $\mathbf{B}^*\mathbf{B}$, respectively.

Proof. We will use the averaging method [4,9]. We introduce the functions

$$\upsilon(t) = \upsilon_n, \quad t \in [t_n, t_{n+1}), \quad u(t) = \int_0^t [\xi(\lambda) - \upsilon(\lambda)]d\lambda$$

Making the Liénad replacement in system (1)

$$\mathbf{x} = \mathbf{y} + \mathbf{b}u \tag{14}$$

we obtain, by virtue of Eq. (7), the equations

$$\dot{\mathbf{y}} = \mathbf{Ay} + \mathbf{a}(\mathbf{y} + \mathbf{b}u) + \mathbf{b}\upsilon + \mathbf{Ab}u \tag{15}$$

$$\sigma = \mathbf{c}^*\mathbf{y} - \kappa u \tag{16}$$

Equation (15) can be represented in the form

$$\dot{\mathbf{y}} = \mathbf{By} + \mathbf{g} \tag{17}$$

$$\mathbf{g} = \mathbf{g}_1 + \mathbf{g}_2 + \mathbf{g}_3$$

$$\mathbf{g}_1 = \mathbf{b}(\upsilon - \varphi) + (\mathbf{Ab} - k\kappa\mathbf{b})u, \quad \mathbf{g}_2 = \mathbf{b}(\varphi - k\sigma), \quad \mathbf{g}_3 = \mathbf{a}(\mathbf{y} + \mathbf{b}u)$$

Consider Lyapunov's function $V(\mathbf{y}) = \mathbf{y}^*\mathbf{Hy}$, where the positive-definite matrix $\mathbf{H}$ is the solution of Eq. (10). The derivative with respect to time of V, taken by virtue of system (15), has the form

$$\dot{V} = -\|\mathbf{y}\|^2 + L_1 + L_2 + L_3; \quad L_i = 2(\mathbf{Hy}, \mathbf{g}_i), \quad i = 1, 2, 3 \tag{18}$$

The following inequalities are obvious

$$\lambda_-\|\mathbf{y}\|^2 \le V \le \lambda_+\|\mathbf{y}\|^2, \quad \|\mathbf{Hy}\|^2 \le \lambda_+ V \tag{19}$$

In view of the second inequality of (19) we have

$$|L_1| \le \varepsilon\lambda_+ V + \frac{1}{\varepsilon}\|\mathbf{g}_1\|^2 \le \varepsilon\lambda_+ V + \frac{2}{\varepsilon}(\upsilon - \varphi)^2\|\mathbf{b}\|^2 + \frac{2}{\varepsilon}u^2\|\mathbf{Ab} - k\kappa\mathbf{b}\|^2 \tag{20}$$

where ε is a positive parameter, the choice of which will be discussed below.

We will first assume that condition (9) is satisfied for all σ. In view of this condition and of Eq. (16) we have the limit

$$|\varphi(\sigma) - k\sigma| \le \varphi_+\sigma^2/2 \le \varphi_+(\|\mathbf{c}\|^2\|\mathbf{y}\|^2 + \kappa^2 u^2)$$

Hence, according to the second inequality of (19), the following relation holds

$$|L_1| \le 2\sqrt{\lambda_+ V}\varphi_+\|\mathbf{b}\|[\|\mathbf{c}\|^2 V/\lambda_- + \kappa^2 u^2] \tag{21}$$

Finally, from the second inequality of (19) we obtain the inequality

$$|L_3| \le 2\sqrt{\lambda_+ V}\|\mathbf{a}\|$$

From this inequality and the second inequality of (19) we obtain the limit

$$|L_3| \le \sqrt{\lambda_+}\mu(\mathbf{y}+\mathbf{b}u)[V + 2(\|\mathbf{y}\|^2 + \|\mathbf{b}\|^2 u^2)] \tag{22}$$

where

$$\mu(\mathbf{y}+\mathbf{b}u) = \|\mathbf{a}(\mathbf{y}+\mathbf{b}u)\|/\|\mathbf{y}+\mathbf{b}u\|$$

By virtue of relations (18)–(22) we have the inequality

$$\begin{gathered}\dot V \le \left[\varepsilon\lambda_+ - \frac{1}{\lambda_+} + \sqrt{\lambda_+}\mu(\mathbf{y}+\mathbf{b}u)\right]V + \frac{2}{\varepsilon}[\|\mathbf{b}\|^2(\upsilon-\varphi)^2 + \|\mathbf{A}\mathbf{b} - k\kappa\mathbf{b}\|^2 u^2] + \\ + 2\sqrt{\lambda_+}\left\{\frac{1}{\lambda_-}\|\mathbf{b}\|\varphi_+\|\mathbf{c}\|^2 V^{3/2} + \|\mathbf{b}\|\varphi_+\kappa^2 u^2\sqrt{V} + \mu(\mathbf{y}+\mathbf{b}u)[\|\mathbf{y}\|^2 + \|\mathbf{b}\|^2 u^2]\right\}\end{gathered} \tag{23}$$

Using property (9) and Eq (16), we obtain

$$\begin{gathered}\Phi \doteq \int_{t_n}^{t_{n+1}} (\upsilon(t) - \varphi(\sigma(t)))^2 dt \le l^2 \int_{t_n}^{t_{n+1}} |\sigma(\tilde t_n) - \sigma(t)|^2 dt \le \\ \le 2l^2 \int_{t_n}^{t_{n+1}} [|\mathbf{c}^*\mathbf{y}(\tilde t_n) - \mathbf{c}^*\mathbf{y}(t)|^2 + \kappa^2|u(\tilde t_n) - u(t)|^2]dt\end{gathered}$$

It is well known [4, 9], that for any absolutely continuous function $\zeta(t)$ with $\dot\zeta \in L_2[\alpha, \beta]$ and any $\tilde t_n \in [\alpha, \beta]$ the Wirtinger inequality holds, namely

$$\int_\alpha^\beta |\zeta(\tilde t) - \zeta(t)|^2 dt \le \frac{4(\beta-\alpha)^2}{\pi^2}\int_\alpha^\beta |\dot\zeta(t)|^2 dt$$

Hence

$$\Phi \le \frac{8l^2T^2}{\pi^2}\int_{t_n}^{t_{n+1}} |\mathbf{c}^*\dot{\mathbf{y}}|^2 dt + 4l^2\kappa^2 \int_{t_n}^{t_{n+1}} [u^2(\tilde t_n) + u^2(t)]dt$$

Substituting expression (15) into this inequality and using the limit [4, 9]

$$|u(t)| \le T|\upsilon(t)| \tag{24}$$

we obtain a chain of relations

$$\begin{gathered}\Phi \le \frac{8l^2T^2}{\pi^2}\int_{t_n}^{t_{n+1}} |\mathbf{c}^*\mathbf{A}\mathbf{y} - \kappa\upsilon + \kappa_1 u + \mathbf{c}^*\mathbf{a}|^2 dt + 8l^2\kappa^2T^2 \int_{t_n}^{t_{n+1}} \upsilon^2(t)dt \le \\ \le \int_{t_n}^{t_{n+1}} \left\{\frac{8l^2T^2}{\pi^2}[\|\mathbf{c}^*\mathbf{A}\|\|\mathbf{y}\| + (|\kappa| + |\kappa_1|T)|\upsilon| + \|\mathbf{c}\|\|\mathbf{a}\|]^2 + 8l^2\kappa^2T^2\upsilon^2\right\}dt \le \\ \le \int_{t_n}^{t_{n+1}} \left\{\frac{24l^2T^2}{\pi^2}[\|\mathbf{c}^*\mathbf{A}\|^2\|\mathbf{y}\|^2 + (|\kappa| + |\kappa_1|T)^2\upsilon^2 + \|\mathbf{c}\|^2\|\mathbf{a}\|^2] + 8l^2\kappa^2T^2\upsilon^2\right\}dt\end{gathered}$$

We will assume that

$$\mu(\mathbf{y} + \mathbf{b}u) < \delta_1 \tag{25}$$

Then

$$\|\mathbf{a}(\mathbf{y} + \mathbf{b}u)\|^2 \le 2\delta_1^2(\|\mathbf{y}\|^2 + \|\mathbf{b}\|^2 u^2)$$

and the limit obtained for Φ can be represented in the form

$$\Phi \le c_1(\delta_1)T^2 Y + c_2(\delta_1)T^2 X \tag{26}$$

where

$$Y = \int_{t_n}^{t_{n=1}} \|\mathbf{y}(t)\|^2 dt, \quad X = \int_{t_n}^{t_{n+1}} \upsilon^2(t)dt$$

$$c_1(\delta_1) = \frac{24l^2}{\pi^2}[\|\mathbf{c}^*\mathbf{A}\|^2 + 2\delta_1^2\|\mathbf{c}\|^2]$$

$$c_2(\delta_1) = \frac{24l^2}{\pi^2}[(|\kappa| + |\kappa_1|T)^2 + 2\delta_1^2\|\mathbf{b}\|^2\|\mathbf{c}\|^2\kappa^2T^2] + 8l^2\kappa^2$$

We will estimate X in terms of Y. The following inequalities are obtained from relations (9) and (25)

$$|\varphi(\sigma)| \le l|\mathbf{c}^*\mathbf{y} - \kappa u|, \quad \varphi^2(\sigma) \le 2l^2(\|\mathbf{c}\|^2\|\mathbf{y}\|^2 + \kappa^2T^2\upsilon^2)$$

Since $\nu = \varphi + (\upsilon - \varphi)$, we have

$$\upsilon^2 \le 2\varphi^2 + 2(\upsilon + \varphi)^2 \le 4l^2\|\mathbf{c}\|^2\|\mathbf{y}\|^2 + 4l^2\kappa^2T^2\upsilon^2 + 2(\upsilon - \varphi)^2$$

Hence the following relation holds

$$X \le 4l^2\|\mathbf{c}\|^2 Y + 4l^2\kappa^2T^2X + 2\Phi$$

Estimating the right-hand side of this inequality and using inequality (26), we arrive at the limit

$$X \le [4l^2\|\mathbf{c}\|^2 + 2c_1(\delta_1)T^2]Y + [4l^2\kappa^2T^2 + 2c_2(\delta_1)T^2]X \tag{27}$$

If the coefficient of X on the right-hand side of this inequality is less than unity (which will be the case for sufficiently small δ_1 in view of assumption (11)), we obtain the following relation from limit (27)

$$X \le d_1(\delta_1)Y, \quad d_1(\delta_1) = \frac{4l^2\|\mathbf{c}\|^2 + 2c_1(\delta_1)T^2}{1 - [4l^2\kappa^2 + 2c_2(\delta_1)]T^2} \tag{28}$$

From relation (26) and (28) we obtain the inequality

$$\Phi \le d_2(\delta_1)Y, \quad d_2(\delta_1) = [c_1(\delta_1) + c_2(\delta_1)d_1(\delta_1)]T^2 \tag{29}$$

Consider the region $D = \{\mathbf{y} : V(\mathbf{y}) \le \delta^2\}$. In this region, according to limit (23) and (24), the following relation holds

$$\dot{V} \le -\nu(\delta, \delta_1)V + F \tag{30}$$

$$\nu(\delta, \delta_1) = \frac{1}{\lambda_+} - \varepsilon\lambda_+ - \frac{2}{\lambda_-}\sqrt{\lambda_+}\|\mathbf{b}\|\varphi_+\|\mathbf{c}\|^2\delta - \sqrt{\lambda_+}\delta_1$$

$$F = \frac{2}{\varepsilon}\|\mathbf{b}\|^2(\upsilon - \varphi)^2 + \frac{2T^2}{\varepsilon}\|\mathbf{A}\mathbf{b} - k\kappa\mathbf{b}\|^2\upsilon^2 + 2T^2\sqrt{\lambda_+}\|\mathbf{b}\|\varphi_+\kappa^2\upsilon^2\delta + 2\sqrt{\lambda_+}\delta_1[\|\mathbf{y}\|^2 + T^2\|\mathbf{b}\|^2\upsilon^2]$$

In view of relations (28) and (29) we have the limit

$$\int_{t_n}^{t_{n+1}} F dt \le d_3(\delta, \delta_1) \int_{t_n}^{t_{n+1}} V dt \tag{31}$$

$$d_3(\delta, \delta_1) = \frac{1}{\lambda_-}\Big[\frac{2T^2}{\varepsilon}\|\mathbf{b}\|^2 d_2(\delta_1) + \frac{2T^2}{\varepsilon}\|\mathbf{A}\mathbf{b} - k\kappa\mathbf{b}\|^2 d_1(\delta_1) +$$

$$+ 2T^2\sqrt{\lambda_+}\|\mathbf{b}\|\varphi_+\kappa^2\delta d_1(\delta_1) + 2\sqrt{\lambda_+}\delta_1 + 2\sqrt{\lambda_+}\delta_1 T^2\|\mathbf{b}\|^2 d_1(\delta_1)\Big]$$

From relations (30) and (31) we obtain the inequalities

$$V(\mathbf{y}(t_{n+1})) - V(\mathbf{y}(t_n)) \le -\lambda(\delta, \delta_1)\int_{t_n}^{t_{n+1}} V(\mathbf{y}(t))dt$$

$$\lambda(\delta, \delta_1) = \nu(\delta, \delta_1) - d_3(\delta, \delta_1)$$

Summing these inequalities over n from 0 to $N - 1$, we obtain the relation

$$V(\mathbf{y}(t_N)) + \lambda(\delta, \delta_1)\int_0^{t_N} V(\mathbf{y}(t))dt \le V(\mathbf{y}(0)) \tag{32}$$

We require that $\lambda(0, 0) > 0$. We have

$$\lambda(0, 0) = \nu(0, 0) - d_3(0, 0) = \frac{1}{\lambda_+} - \varepsilon\lambda_+ - \frac{T^2}{\varepsilon}p$$

Hence, the relation $\lambda(0, 0) > 0$ is equivalent to the inequality

$$\lambda_+^2\varepsilon^2 - \varepsilon + T^2 p\lambda_+ < 0$$

which, in view of condition (12), is satisfied when

$$\varepsilon_- < \varepsilon < \varepsilon_+, \quad \varepsilon_\pm = (1 \pm \sqrt{1 - 4\lambda_+^3 pT^2})/(2\lambda_+^2)$$

It is obvious that $\mu(\delta, \delta_1) > 0$ for sufficiently small δ and δ_1. It follows from inequality (32) that $\mathbf{y}(t_n) \in D$ for all n if $y(0) \in D$.

It can be shown that $\mathbf{y}(t) \in D$ for all $t > 0$ if the quantity $\|y(0)\|$ is sufficiently small. Since $|\varphi(\sigma)| \leqslant l|\sigma|$, we have

$$|\upsilon_n| \le l|\sigma(\tilde{t}_n)| \le l\|\mathbf{c}\|\delta_2 + l|\kappa||\upsilon_n|T, \quad \delta_2 = \max_{t \in [t_n, t_{n+1}]}\|\mathbf{y}(t)\|$$

Hence, in view of condition (13), we obtain the limit

$$|\upsilon_n| \le d_4\delta_2, \quad d_4 = l\|\mathbf{c}\|/(1 - l|\kappa|T) \tag{33}$$

Hence, in inequality (25)

$$\lim_{\delta_2 \to 0}\delta_1 = 0 \tag{34}$$

Integrating Eq. (15), we obtain the representation

$$\mathbf{y}(t) = \exp(\mathbf{A}(t - t_n))\mathbf{y}(t_n) +$$

$$+ \int_{t_n}^{t}\exp(\mathbf{A}(t - \lambda))\{\mathbf{a}(\mathbf{y}(\lambda) + \mathbf{b}u(\lambda)) + \mathbf{b}\upsilon(\lambda) + \mathbf{A}\mathbf{b}u(\lambda)\}d\lambda$$

Hence, for $t_n < t \leqslant t_{n+1}$, by virtue of limit (23) and the first inequality of (19), we have the limit

$$\begin{aligned} &\delta_2 \le \exp(\|\mathbf{A}\|T)\sqrt{V(\mathbf{y}(0))/\lambda_-} + \delta_2 s \\ &s = \{(\|\mathbf{b}\| + \|\mathbf{Ab}\|T)d_4 + (1 + \|\mathbf{b}\|Td_4)\delta_1\}\exp(\|\mathbf{A}\|T) \end{aligned} \tag{35}$$

If

$$s < 1 \tag{36}$$

from inequality (35) we have the limit

$$\delta_2 \le \exp(\|\mathbf{A}\|T)\sqrt{V(\mathbf{y}(0))}/[(1-s)\sqrt{\lambda_-}]$$

and consequently $\mathbf{y}(t) \in D$ for sufficiently small $\|\mathbf{y}(0)\|$. In view of condition (13) and property (34), inequality (36) is satisfied for sufficiently small δ. Since $\mathbf{x}(0) = \mathbf{y}(0)$ and, by virtue of relations (14), (24) and (33)

$$\max_{t \in [t_n, t_{n+1}]} \|\mathbf{x}(t)\| \le \delta_2(1 + |\kappa|Td_4)$$

the Lyapunov stability of the equilibrium state $x = 0$ is proved.

We will show that $\|\mathbf{x}(t)\| \to 0$ when $t \to +\infty$ if $\|\mathbf{x}(0)\|$ is sufficiently small. According to relation (32), $\|\mathbf{y}(t)\| \in L_2[0, +\infty)$. Since, by virtue of Eq. (15), the quantity $\|\dot{\mathbf{y}}(t)\|$ is uniformly bounded with respect to t, we have $\|\mathbf{y}(t)\| \to 0$ as $t \to +\infty$. Then, in view of relations (14), (24) and (33) $\|\mathbf{x}(t)\| \to 0$ as $t - +\infty$. And the theorem is proved with the additional assumption that conditions (9) are satisfied when $-\infty < \sigma < +\infty$. It can be shown that this assumption is unnecessary. Suppose conditions (9) are satisfied when $|\sigma| \leqslant \sigma_*$. We will denote by $\varphi_*(\sigma)$ the function which is doubly continuously differentiable and satisfies condition (9) for all $\sigma \in (-\infty, +\infty)$, and is identical with $\varphi(\sigma)$ when $|\sigma| \leqslant \sigma_*$. We will determine the operator M_*, which mass $\sigma(t)$ into $\xi_*(t)$ and $\{t_n^*\}$ as follows. Suppose $|\sigma(t)| < \sigma_*$ when $0 \leqslant t < t_*$ and $|\sigma(t_*)| = \sigma_*$. When $t \leqslant t_*$ we have $\xi_*(t) = \zeta(t)$ and $t_n^* = t_n$ when $t_n \leqslant t_*$. Suppose

$$N = \max_{t_m \le t_*} n$$

We then put $t_{n+1}^* = t_n^* + T$ when $n \geqslant N$ and $\xi_*(t) = \varphi_*(\sigma(t_n^*))$ when $t_n^* \leqslant t < t_{n+1}^*$. It is obvious that for $n \geqslant N$ property (4) is satisfied when $\tilde{t}_n^* = t_n^*$. Suppose $\mathbf{x}_*(t)$, which satisfies the condition $\mathbf{x}_*(0) = \mathbf{x}(0)$, is the solution of system (1), in which the operator M is replaced by M_*. Then, in view of what is proved above, $|\mathbf{c}^*\mathbf{x}_*(t)| < \sigma_*$ for all $t > 0$ if the quantity $|\mathbf{x}(0)|$ is sufficiently small. Consequently, in this solution M is identical with M_* and $\mathbf{x}(t) = \mathbf{x}_*(t)$. Hence $|\mathbf{c}^*\mathbf{x}(t)| < \sigma_*$ for all $t > 0$, and the assumption that condition (9) holds for all $\sigma \in (-\infty, +\infty)$ can be removed. The theorem is proved.

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